Applying linear interpolation on grouped data

Vong Jun Yi (vongjy.github.io)

Percentiles

We can estimate the percentile of grouped data using linear interpolation.

Definition

> Linear interpolation is a method of curve fitting using linear polynomials to construct new data points within the range of a discrete set of known data points.

Theorem

> We can estimate the percentile with the formula:
>
> $$P = L + \frac{\frac{RN}{100} - M}{F} C$$
>
> where
>
> L = lower boundary of the class interval containing the percentile
>
> R = percentile rank
>
> N = total frequency of observed data
>
> M = cumulative frequency of observations up to the preceding class
>
> F = frequency of the class interval containing the percentile
>
> C = size of the class interval containing the percentile
>
> P = percentile value

Explanation

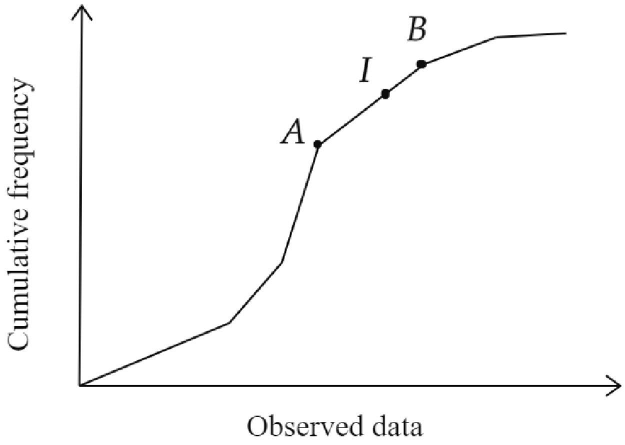
Suppose we plotted a graph of the cumulative frequency against observed data.
Using both the lower bound and upper bound of the class, we have two points $A(L, M)$ and $B(L + C, M + F)$.

We define R_i and R_f as the percentile rank of A and B respectively.

Hence,

$$R_i = \frac{M}{N} \times 100\%$$
$$R_f = \frac{M+F}{N} \times 100\%.$$

Let I be a point on $\overline{AB}$ where R is its percentile rank (see Figure 1).



Note that I divides $\overline{AB}$ in the ratio $(R - R_i) : (R_f - R)$. Hence, the coordinates of I are

$$\begin{aligned} I(x, y) &= \left(\frac{nx_1 + mx_2}{m+n}, \frac{ny_1 + my_2}{m+n} \right) \\ &= \left(\frac{(R_f - R)L + (R - R_i)(L + C)}{R_f - R_i}, \frac{(R_f - R)M + (R - R_i)(M + F)}{R_f - R_i} \right) \\ &= \left(L + \left(\frac{R - R_i}{R_f - R_i} \right) C, M + \left(\frac{R - R_i}{R_f - R_i} \right) F \right) \end{aligned}$$

The x-coordinate of I is the percentile value:

$$\begin{aligned} P &= L + \left(\frac{R - R_i}{R_f - R_i} \right) C \\ &= L + \left(\frac{R - \frac{100M}{N}}{\frac{100}{N} F} \right) C \\ \therefore P &= \boxed{L + \frac{\frac{RN}{100} - M}{F} C} \end{aligned}$$

and we are done. ■

Additional note: The y-coordinate tells us that P is the $\left(\frac{RN}{100} \right)^{\text{th}}$ observation.

One of the limitations of this formula is that it assumes uniform distribution in each class. This may be inaccurate for skewed or heavily concentrated data within classes. At extreme percentiles (near 0 or 100), this inaccuracy may be more apparent.

Nevertheless, if the class sizes for the grouped data were smaller, this inaccuracy decreases as linear interpolation becomes a better approximation of the actual distribution of the observed data.

Suppose we know the actual function of data distribution $f(x)$ (frequency vs data), then

$$\begin{aligned}\int_0^P f(x)\,dx &= \frac{RN}{100}\\ F(P) &= \frac{RN}{100}\\ P &= F^{-1}\left(\frac{RN}{100}\right)\end{aligned}$$

Mean (arithmetic mean)

We first obtain the midpoints of the classes $x_1, x_2, \ldots, x_n$ where $x_i = (a_i + b_i)/2$ for all $i \in (\mathbb{Z}^+ \cap [1, n])$
.
(a_i and b_i are the lower boundary and the upper boundary of the i-th class respectively)

Then, the sample mean $\overline{x}$ can be estimated as follows:

$$\overline{x} = \frac{\sum f_i x_i}{\sum f_i}$$

where f is the class frequency.

Note: This method of computing the sample mean may be inaccurate as it assumes uniform distribution within classes.

Mode

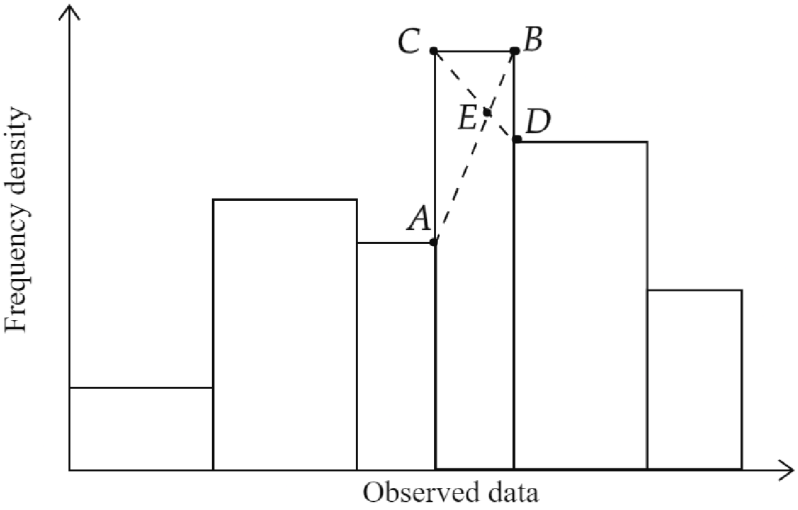
Theorem

> The mode of grouped data can be estimated with the formula:
>
> $$\text{Mode} = L + \frac{f_1 - f_0}{2f_1 - f_0 - f_2}C$$
>
> where
> L = lower boundary of modal class
>
> C = class size/interval
>
> f_0 = frequency density of preceding class
>
> f_1 = frequency density of modal class

> f_2 = frequency density of succeeding class.
>
> Equivalently,
>
> $$\text{Mode} = L + \frac{d_1}{d_1 + d_2} C$$
>
> where $d_1 = f_1 - f_0$ and $d_2 = f_2 - f_1$.

Explanation

Suppose we plotted a histogram for our grouped data and we marked 4 points A, B, C, D on the bar of the modal class (see Figure 2) as follows:



The coordinates of the 4 points are $A(L, f_0)$, $B(L + C, f_1)$, $C(L, f_1)$, $D(L + C, f_2)$.

Let E be the point of intersection of AB and CD. We seek to obtain the mode, which is the x-coordinate of E.

Equation of line AB:

$$y - f_0 = \frac{f_1 - f_0}{C}(x - L) \ldots\ldots\ldots (1)$$

Equation of line CD:

$$y - f_1 = \frac{f_2 - f_1}{C}(x - L) \ldots\ldots\ldots (2)$$

By solving the system of equations above, we obtain

$$\text{Mode} = \boxed{x = L + \frac{f_1 - f_0}{2f_1 - f_0 - f_2} C}$$

and we are done. ■

Subsequently, we can obtain the estimated frequency density of the mode:

$$y = \frac{f_1^2 - f_0 f_2}{2f_1 - f_0 - f_2}.$$

IMPORTANT: The formulae above use f to mean frequency density (NOT FREQUENCY) unless sizes of all classes are equal.

Limitations:

- Assumes uniform distribution within each class. Inaccuracies may occur if the data within classes are heavily skewed.
- Assumes symmetrical modal class.
- May not accurately represent the number of modes if there are multiple modal classes (e.g. unimodal data but multiple modal classes).