Antigraphsketching

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In this article, we will solve an inequality using pure algebra instead of the conventional graph sketching method.

> Given $f(x) = \dfrac{x^2+1}{(x+1)(x-7)}$, find the set of values of x that satisfy $f(x) > \dfrac{1}{f(x)}$.

We first note that $f(x) > 0 \iff x < -1 \lor x > 7$ and $f(x) < 0 \iff -1 < x < 7$ by sign test.

Case #1: $f(x) > 0$

$$\begin{aligned} & f(x) > \frac{1}{f(x)} \\ \implies & (f(x))^2 - 1 > 0 \\ \iff & (f(x)-1)(f(x)+1) > 0 \end{aligned}$$

From the inequality above, $f(x) < -1$ (reject since $f(x) > 0$) and $f(x) > 1$.

When $f(x) > 1$,

$$\begin{aligned} & \frac{x^2+1}{(x+1)(x-7)} > 1 \\ \iff & \frac{6x+8}{(x+1)(x-7)} > 0 \\ \iff & \frac{x+\frac{4}{3}}{(x+1)(x-7)} > 0 \end{aligned}$$

By sign test, $-\dfrac{4}{3} < x < -1$ and $x > 7$.

Case #2: $f(x) < 0$

$$\begin{aligned} & f(x) > \frac{1}{f(x)} \\ \implies & (f(x))^2 - 1 < 0 \\ \iff & (f(x)-1)(f(x)+1) < 0 \end{aligned}$$

From the inequality above, $-1 < f(x) < 0$.

This gives

$$-1 < \frac{x^2+1}{(x+1)(x-7)} < 0$$

This case works for $-1 < x < 7$, so $(x+1)(x-7) < 0$.

We split the inequality into two parts again.

$$\begin{aligned} &\frac{x^2+1}{(x+1)(x-7)} < 0 \\ \iff &x^2+1 > 0 \\ \iff &x^2 > -1 \\ \implies &x \in \mathbb{R} \end{aligned}$$

$$\begin{aligned} &\frac{x^2+1}{(x+1)(x-7)} > -1 \\ \implies &x^2+1 < 6x+7-x^2 \\ \iff &x^2-3x-3 < 0 \\ \implies &\frac{3-\sqrt{21}}{2} < x < \frac{3+\sqrt{21}}{2} \end{aligned}$$

From case #2, we conclude $\frac{3-\sqrt{21}}{2} < x < \frac{3+\sqrt{21}}{2}$.

Thus,

$$x \in \boxed{\left(-\frac{4}{3}, 1\right) \cup \left(\frac{3-\sqrt{21}}{2}, \frac{3+\sqrt{21}}{2}\right) \cup (7, \infty)}.$$

Link to graph