IMONST 1 2024 Senior
Problems and Solutions

Vong Jun Yi

1 Problems

All problem credits go to IMO Committee Malaysia.

Section A

Each problem in this section is worth **1 point**. For each of the following statements, determine whether it is true or false.

Problem 1 The sum of an odd number of odd numbers will always be an odd number.

Problem 2 $\sqrt{a^2+b^2} = a+b$

Problem 3 The equation x^2+7x+k can have equal integer roots for some integer k.

Problem 4 $0/0! \in \mathbb{R}$

Problem 5 If $a > b$ and $b^2 > a^2$, then both a and b must be negative.

Problem 6 There are infinitely many rational numbers between e and π.

Problem 7 In Euclidean geometry, parallel lines will never meet.

Problem 8 There are exactly 6 possible last digits for a perfect square.

Problem 9 The difference between a positive integer and its reverse will always be divisible by 9.

Problem 10 If a divides c and b divides c, then ab divides c.

Section B

Each problem in this section is worth **3 points**.

Problem 1 When 390 is divided by either 7 or 11, it gives the same non-zero remainder. How many integers from 1 to 2000 have this property?

Problem 2 Let $PQRS$ be a convex quadrilateral. The points T, U, V and W are the midpoints of PQ, QR, RS and SP respectively. If $\angle TUV = 70^\circ$, find $\angle UVW$, in degrees.

Problem 3 What is the value of $100^2 - 97^2 + 94^2 - 91^2 + \cdots + 4^2 - 1^2$?

Problem 4 How many distinct ways are there to arrange the letters of the word 'MALAYSIA' in a circle?

Problem 5 If we multiply all even integers from 22 to 222, what are the last 2 digits of the product?

Problem 6 A rhombus is inscribed inside a circle, and its perimeter is 24. If the area of the circle is $A\pi$, find A.

Problem 7 Find the largest integer N such that 2^N divides $3^{16} - 1$.

Problem 8 Find the smallest positive multiple of 198 such that 1s and 2s are its only digits.

Problem 9 Let $\triangle ABC$ be an equilateral triangle, and D is a point outside of the triangle such that $\angle ADB$ is 90°. If AM is the median, find $\angle ADM$, in degrees.

Problem 10 Let $S_k = 1 + 2 + \cdots + k$. How many values of k are there such that S_k is a prime number?

Section C

Each problem in this section is worth **6 points**.

Problem 1 Let $ABCD$ be an orthodiagonal quadrilateral circumscribed around a circle, with $BC = 10, CD = 20$, and $AD = 30$. Find AB.

Problem 2 A number N has 240 divisors and 60 of them are odd. If N is divisible by 2^k, what is the largest possible value of k?

Problem 3 Let a and b be positive integers. Find $a + b$ if $a^2 + a = b^4 + b^3 + b^2 + b$.

Problem 4 In a country, there are 2024 cities connected by non-crossing roads. A citizen of the country can travel between any two cities along the road segments, using each at most once, in exactly one way. How many road segments are there in the country?

Problem 5 Let $N = \overline{aabcaabc\ldots aabc}$ be a 100-digit number where a, b, c are non-zero digits. Given that N is divisible by the sum of its digits, find the last 2 digits of N.

Problem 6 Given a triangle ABC with area 40 and perimeter 20, such that its incircle is centered at I. Point D is drawn on the triangle's circumcircle so that AD passes through I and intersects side BC at E. If $CD = 10$, find DE.

Problem 7 Given that in triangle PQR, $PQ = PR$ and $\angle QPR = 120°$. S is a point on QR such that $QS = 20$ and $SR = 40$. Find PS.

Problem 8 What is the smallest k such that among any k consecutive positive integers, there is always a number with a digit sum (in base 10) that is divisible by 11?

Problem 9 Given $a^7 + a^6 + a^5 + a^4 + a^3 + a^2 + a^1 + a^0 = 0$. Find $a^{4048} + a^{2024} + 1$.

Problem 10 Sean wants to create a word using the letters from 'ABC' only. He may use the same letters as much as he wants. How many 5-letter words can he make such that no two letters B are beside each other?

2 Solutions

Section A

Problem 1 The sum of an odd number of odd numbers will always be an odd number.

Solution True. The sum of two odd numbers always gives an even number. The sum of two even numbers also always gives an even number. If we had $(2n + 1)$ odd numbers O_k, we can consider pairs of odd numbers from $2n$ of them, where each pair gives an even sum.

$$S = O_1 + O_2 + \cdots + O_{2n-1} + O_{2n} + O_{2n+1}$$

For example, $O_{2k-1} + O_{2k} = E_k$ for $1 \leq k \leq n$.

$$S = E_1 + E_2 + \cdots + E_{n-1} + E_n + O_{2n+1}$$

This gives n even numbers, whose sum is also an even number. Define $E = E_1 + E_2 + \cdots + E_k$.

$$S = E + O_{2n+1}$$

Since an even number and an odd number always gives an odd sum, S is odd.

■

Problem 2 $\sqrt{a^2 + b^2} = a + b$

Solution False. Squaring both sides gives $a^2 + b^2 = a^2 + 2ab + b^2$ which is false for $a \neq 0$ and $b \neq 0$.

■

Problem 3 The equation $x^2 + 7x + k$ can have equal integer roots for some integer k.

Solution False. If the equation has equal roots for some integer k, the discriminant Δ is equal to 0.

$$7^2 - 4k = 0$$
$$k = \frac{49}{4}$$

However, $k = 49/4$ is not an integer.

■

Problem 4 $0/0! \in \mathbb{R}$

Solution True. $0/0! = 0/1 = 0$, and 0 is a real number.

■

Problem 5 If $a > b$ and $b^2 > a^2$, then both a and b must be negative.

Solution False. Consider the example $(a, b) = (1, -2)$. We have $1 > -2$ and $(-2)^2 = 4 > 1 = 1^2$, but a is not negative.

■

Problem 6 There are infinitely many rational numbers between e and π.

Solution True. This is a consequence of the density of rational numbers on the real number line. Rational numbers are dense in $\mathbb{R}$, which means that for any two real numbers a and b with $a < b$, there exists a rational number q such that $a < q < b$. Not only that, but you can actually find infinitely any such rational numbers between a and b.

■

Problem 7 In Euclidean geometry, parallel lines will never meet.

Solution True. This is a result from the 5^{th} postulate in Euclid's *Elements*:

- "If a line segment intersects two straight lines forming two interior angles on the same side that are less than two right angles, then the two lines, if extended indefinitely, meet on that side on which the angles sum to less than two right angles."

From this, we can deduce that parallel lines do not meet since they cannot form two interior angles on the same side that are less than two right angles (i.e. one obtuse, one acute). Alternatively, Playfair's axiom may be considered too since it is mathematically equivalent to the parallel postulate.

- "In a plane, given a line and a point not on it, at most one line parallel to the given line can be drawn through the point."

■

Problem 8 There are exactly 6 possible last digits for a perfect square.

Solution True. Squaring numbers with the last digit $0, 1, 2, 3, 4, 5, 6, 7, 8, 9$ will yield numbers with the last digit $0, 1, 4, 9, 6, 5, 6, 9, 4, 1$ respectively.

Hence, $0, 1, 4, 5, 6$ and 9 are the only possible last digits for a perfect square (in base 10).

■

Problem 9 The difference between a positive integer and its reverse will always be divisible by 9.

Solution True. Let $N = \overline{a_k a_{k-1} \cdots a_1 a_0}$ and its reverse $N' = \overline{a_0 a_1 \cdots a_{k-1} a_k}$. By considering their difference,

$$\begin{aligned} N - N' &= (10^k a_k + 10^{k-1} a_{k-1} + \cdots + 10a_1 + a_0) - (a_k + 10a_{k-1} + \cdots + 10^{k-1} a_1 + 10^k a_0) \\ &\equiv (1^k a_k + 1^{k-1} a_{k-1} + \cdots + a_1 + a_0) - (a_k + a_{k-1} + \cdots + 1^{k-1} a_1 + 1^k a_0) \\ &\equiv (a_k + a_{k-1} + \cdots + a_1 + a_0) - (a_k + a_{k-1} + \cdots + a_1 + a_0) \\ N - N' &\equiv 0 \pmod 9 \end{aligned}$$

Hence, $(N - N')$ is always divisible by 9.

■

Problem 10 If a divides c and b divides c, then ab divides c.

Solution False. Take $(a, b, c) = (4, 6, 12)$ as an example. $4 \mid 12$ and $6 \mid 12$, but $24 \nmid 12$.

■

Section B

Problem 1 When 390 is divided by either 7 or 11, it gives the same non-zero remainder. How many integers from 1 to 2000 have this property?

Solution We want to find number of integers n where $1 \leq n \leq 2000$ that satisfy the following properties:

$$\begin{aligned} n &\equiv k \pmod 7 \\ n &\equiv k \pmod{11} \end{aligned}$$

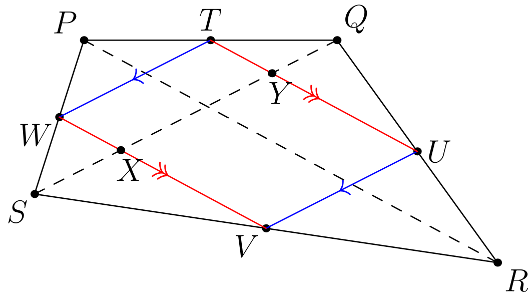
where $0 \leq k \leq 6$. This is equivalent to looking for integers with the form $n = 77q + k$. We only have to consider values of q not exceeding $\left\lfloor \frac{2000}{77} \right\rfloor$.
When $q = 0$, $k = 1, 2, 3, 4, 5, 6$.
When $1 \leq q \leq 25$, $k = 0, 1, 2, 3, 4, 5, 6$

In total, there are $6 + 25(7) = \boxed{181}$ such integers.

■

Problem 2 Let $PQRS$ be a convex quadrilateral. The points T, U, V and W are the midpoints of PQ, QR, RS and SP respectively. If $\angle TUV = 70°$, find $\angle UVW$, in degrees.

Solution



From the Midpoint Theorem, $UV||QS||TW$ and $UT||PR||VW$. $TUVW$ forms a parllelogram. Hence,

$$\angle TUV + \angle UVW = 180°$$
$$\angle UVW = \boxed{110°}$$

A more straightforward way to approach the problem is to consider Varignon's Theorem.

■

Problem 3 What is the value of $100^2 - 97^2 + 94^2 - 91^2 + \cdots + 4^2 - 1^2$?

Solution Let $S = 100^2 - 97^2 + 94^2 - 91^2 + \cdots + 4^2 - 1^2$. We shall split the sum into two parts (positive and negative), so

$$P := \sum_{k=0}^{16} (6k+4)^2 = 4^2 + 10^2 + \cdots + 94^2 + 100^2$$

$$N := \sum_{k=0}^{16} (6k+1)^2 = 1^2 + 7^2 + \cdots + 91^2 + 97^2$$

Then, we can rewrite S as follows:

$$\begin{aligned} S &= P - N \\ &= \sum_{k=0}^{16}(6k+4)^2 - \sum_{k=0}^{16}(6k+1)^2 \\ &= \sum_{k=0}^{16}((6k+4)^2 - (6k+1)^2) \\ &= \sum_{k=0}^{16} 3(12k+5) \\ &= 36\sum_{k=0}^{16} k + 15(17) \\ &= 36 \cdot \frac{(16)(17)}{2} + 255 \\ S &= \boxed{5151} \end{aligned}$$

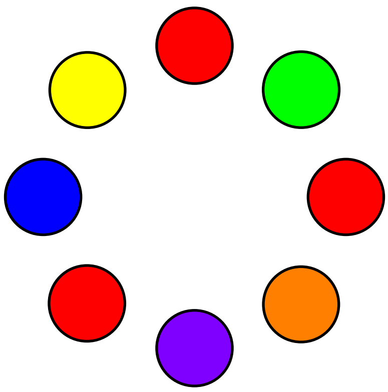
■

Problem 4 How many distinct ways are there to arrange the letters of the word 'MALAYSIA' in a circle?

Solution In general, the number of k-ary necklaces (k distinct beads) of length n that can be formed is given by

$$\# \text{ of necklaces} = \frac{1}{n} \sum_{d \mid \gcd(r_1, r_2, \ldots, r_k)} \phi(d) \binom{n/d}{r_1/d, r_2/d, \ldots, r_k/d}$$

where r_i is the multiplicity of each type of bead for $1 \le i \le k$. This is a result derived from Pólya enumeration theorem.



This problem is equivalent to finding the number of necklaces of length 8 that can be made with 6 different beads, where $(r_1, r_2, r_3, r_4, r_5, r_6) = (3, 1, 1, 1, 1, 1)$. This simplifies to

$$\begin{aligned}\text{\# of arrangements} &= \frac{1}{8}\binom{8}{3,1,1,1,1,1} \\ &= \frac{1}{8}\cdot\frac{8!}{3!} \\ &= \boxed{840}\end{aligned}$$

■

Problem 5 If we multiply all even integers from 22 to 222, what are the last 2 digits of the product?

Solution Let $P = 22 \cdot 24 \cdot 26 \cdots 218 \cdot 220 \cdot 222$.

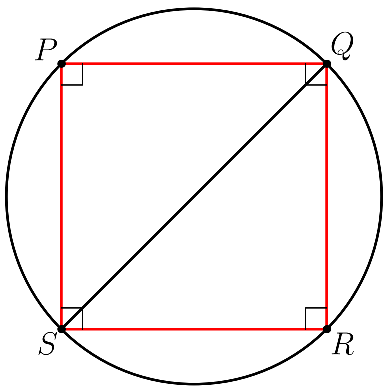
$$\begin{aligned}P &\equiv 22 \cdot 24 \cdots 96 \cdot 98 \cdot 100 \cdot 102 \cdot 104 \cdots 196 \cdot 198 \cdot 200 \cdot 202 \cdot 204 \cdots 220 \cdot 222 \\ &\equiv 22 \cdot 24 \cdots 96 \cdot 98 \cdot 0 \cdot 2 \cdot 4 \cdots 96 \cdot 98 \cdot 0 \cdot 2 \cdot 4 \cdots 20 \cdot 22 \\ &\equiv 0 \pmod{100}\end{aligned}$$

The last two digits are $\boxed{00}$.

■

Problem 6 A rhombus is inscribed inside a circle, and its perimeter is 24. If the area of the circle is $A\pi$, find A.

Solution



Suppose we have a rhombus $PQRS$ inscribed in a circle. Since $PQRS$ is a cyclic quadrilateral, $\angle Q + \angle S = 180°$, however $\angle Q = \angle S$. Hence, $2\angle Q = 180° \iff \angle Q = \angle S = 90°$.

By a similar argument, $\angle P = \angle R = 90°$ Since $PQRS$ is a rhombus, $PQ = QR = RS = SP$. This means $PQRS$ is also a square.

This means a diagonal of the square forms the diameter of the circle. From the Pythagorean thoerem,

$$2r = \sqrt{6^2 + 6^2}$$
$$r = 3\sqrt{2}$$

By considering the area of a circle,

$$A\pi = \pi r^2 \implies A = r^2 = (3\sqrt{2})^2 = \boxed{18}$$

■

Problem 7 Find the largest integer N such that 2^N divides $3^{16} - 1$.

Solution We can approach the problem using factorisation techniques.

$$\begin{aligned} 3^{16} - 1 &= (3^8 + 1)(3^4 + 1)(3^2 + 1)(3 + 1)(3 - 1) \\ &= 6562 \cdot 82 \cdot 10 \cdot 4 \cdot 2 \\ &= (2 \cdot 3281)(2 \cdot 41)(2 \cdot 5)(2^2)(2) \\ &= 2^6 \cdot 3281 \cdot 41 \cdot 5 \end{aligned}$$

$3281 \cdot 41 \cdot 5$ cannot be divided by 2 any further without leaving a remainder, so we are done. The answer is $N = \boxed{6}$.

■

Problem 8 Find the smallest positive multiple of 198 such that 1s and 2s are its only digits.

Solution We can begin expressing 198 as $2 \times 9 \times 11$, so we can consider the following divisibility rules:

(a) If a number is divisible by 2, its last digit must be even.

(b) If a number is divisible by 9, its digit sum must be divisible by 9 too.

(c) If a number is divisible by 11, its alternating digit sum must be divisible by 11.

From (a), it is trivial that the last digit is 2. The multiple we are looking for is at least 5 digits long, since 12222 is the smallest sum that gives a digit sum divisible by 9.
Suppose the multiple we are looking for is $N = \overline{a_1 a_2 \cdots a_{n-1} 2}$, where a_i is the i-th digit. Then, the digit sum and alternating digit sum can be expressed as follows:

$$a_1 + a_2 + \cdots + a_{n-1} + 2 = 9k \quad (1)$$
$$a_1 - a_2 + \cdots + (-1)^{n-1} a_{n-1} + 2(-1)^n = 11m \quad (2)$$

By adding (1) and (2) together, we obtain

$$2a_1 + 2a_3 + 2a_5 + \cdots + (a_{n-1} + (-1)^{n-1}a_{n-1}) + (2 + 2(-1)^n) = 9k + 11m$$

The LHS is even, so (k, m) must have the same parity.

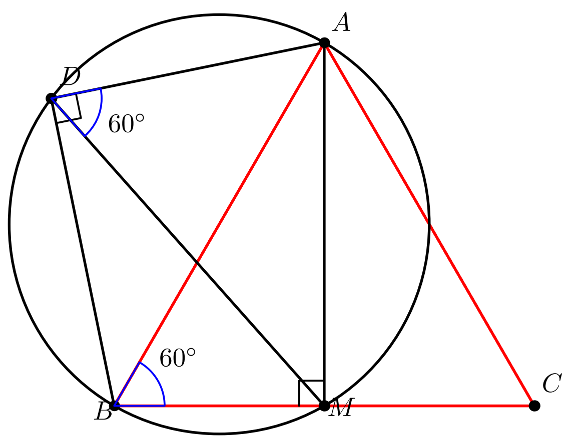
- We see that $k \neq 0$ (digit sum $= 0$) since this will leave us with a degenerate case.
- If $(k, m) = (2, 0)$, the smallest number we can form is $N = 1122222222$ (10 digits long)
- If $m = 1$, the smallest possible number that has an alternating digit sum equal to 11 is $N = \underbrace{21212121212121}_{7\times\ 21}2$, which is 15 digits long. This means considering $m \geq 1$ will only give us increasingly larger values of N.
- If $m = -1$, the smallest possible number that has an alternating digit sum equal to -11 is $N = \underbrace{1212\cdots1212}_{11\times\ 12}$. which is 22 digits long. This means considering $m \leq -1$ will result in increasingly larger values of N.
- When $m = 0$, the smallest number that can be formed when $k = 4$ is $N = \underbrace{22\cdots22}_{18 \text{ digits}}$. This means considering $k \geq 4$ will give us increasingly larger values of N.

Therefore, $(k, m) = (2, 0)$ produces the smallest value of N, which is $N = \boxed{1122222222}$.

■

Problem 9 Let $\triangle ABC$ be an equilateral triangle, and D is a point outside of the triangle such that $\angle ADB$ is $90°$. If AM is the median, find $\angle ADM$, in degrees.

Solution



Since $\angle ADM + \angle AMB = 180°$, $AMBD$ is a cyclic quadrilateral. In a circle, $\angle ABM = \angle ADM = \boxed{60°}$.

■

Problem 10 Let $S_k = 1 + 2 + \cdots + k$. How many values of k are there such that S_k is a prime number?

Solution S_k can be expressed as $k(k+1)/2$. Suppose $S_k = p$ where p is a prime number. Then, we have

$$\frac{k(k+1)}{2} = p$$
$$k(k+1) = 2p$$

If $2 \mid k$, we can write $k = 2m$ where m is a positive integer. Then,

$$2m(2m+1) = 2p$$
$$m(2m+1) = p$$

We need to have one of m and $(2m+1)$ to be equal to 1 since p is prime. If $m = 1$, then $p = 3$. This is valid case. If $2m+1 = 1$, then $m = 0 \implies p = 0$. Contradiction.

If $2|(k+1)$ instead, $k+1 = 2n$

$$(2m-1)(2m) = 2p$$
$$m(2m-1) = 2p$$

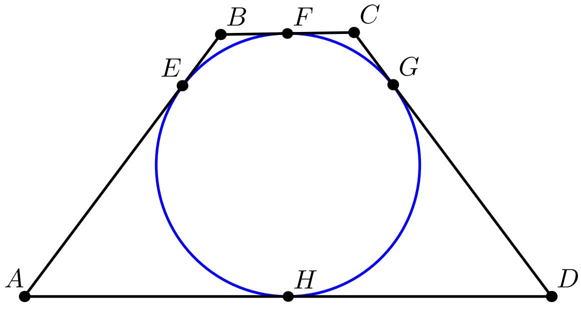
Similarly, we need one of m and $(2m-1)$ to be equal to 1. If $m = 1$, then $p = 1$. Contradiction. If $2m-1 = 1$, then $m = 1 \implies p = 1$. Contradiction. Hence, $k = 2$ is the only value that works. The answer is $\boxed{1}$.

■

Section C

Problem 1 Let $ABCD$ be an orthodiagonal quadrilateral circumscribed around a circle, with $BC = 10, CD = 20$, and $AD = 30$. Find AB.

Solution



Suppose the circle touches AB, BC, CD, and DA at E, F, G, and H respectively. We begin by defining the following:

$$AE = AH = a$$

$$BE + BF = b$$

$$CF + CG = c$$

$$DG + DH = d$$

From this, we can construct a system of equations:

$$a + b = AB \tag{3}$$

$$b + c = 10 \tag{4}$$

$$c + d = 20 \tag{5}$$

$$d + a = 30 \tag{6}$$

By taking (4) + (6) − (5), we obtain $a + b = 20$. Therefore, $AB = \boxed{20}$.

■

Problem 2 A number N has 240 divisors and 60 of them are odd. If N is divisible by 2^k, what is the largest possible value of k?

Solution Suppose the prime factorisation of N is $N = 2^{e_1} p_2^{e_2} p_3^{e_3} \cdots$. In total, the number of divisors of N is

$$(e_1 + 1)(e_2 + 1)(e_3 + 1) \cdots = 240$$

The number of odd divisors can be expressed as

$$(e_2 + 1)(e_3 + 1) \cdots = 60$$

This means $e_1 + 1 = 240/60 = 4 \implies e_1 = 3$. Therefore, $k_{\max} = e_1 = \boxed{3}$.

■

Problem 3 Let a and b be positive integers. Find $a + b$ if $a^2 + a = b^4 + b^3 + b^2 + b$.

Solution By testing small values of b, we see that $b = 2$ works.

$$\begin{aligned} a^2 + a &= 2^4 + 2^3 + 2^2 + 2 \\ &= 30 \\ a^2 + a - 30 &= 0 \\ (a + 6)(a - 5) &= 0 \end{aligned}$$

Since a is positive, $a = 5$, which gives us $a + b = \boxed{7}$.

■

Problem 4 In a country, there are 2024 cities connected by non-crossing roads. A citizen of the country can travel between any two cities along the road segments, using each at most once, in exactly one way. How many road segments are there in the country?

Solution Since there is only one path between any 2 cities, there cannot be cycles or multiple edges in the graph. The graph is a tree. We can treat each city as a vertex, so there are 2024 vertices and $2024 - 1 = \boxed{2023}$ edges (or road segments) only.

■

Problem 5 Let $N = \overline{aabcaabc\ldots aabc}$ be a 100-digit number where a, b, c are non-zero digits. Given that N is divisible by the sum of its digits, find the last 2 digits of N.

Solution The digit sum of N is $50a + 25b + 25c = 25(2a + b + c)$. If we consider N modulo 25,

$$\begin{aligned} N &\equiv \overline{aabcaabc\cdots aabc} \\ &\equiv \overline{aabc} \times 10^{96} + \overline{aabc} \times 10^{92} + \cdots + \overline{aabc} \times 10^{4} + \overline{aabc} \\ &\equiv \overline{aabc} \pmod{25} \end{aligned}$$

Since N is divisible by its digit sum, N is divisible by 25 too, i.e. $\overline{aabc} \equiv 0 \pmod{25}$. Since $25 \mid \overline{aabc}$, $\overline{bc}$ can be $25, 50, 75$, or 00. However, b and c are non-zero, so $\overline{bc} = 25$ or 75 are the only possible cases we can consider.
If we divide N by 25,

$$\frac{N}{25} = \underbrace{\overline{aabc} \times 10^{95} \times 4 + \overline{aabc} \times 10^{91} \times 4 + \cdots + \overline{aabc} \times 10^{3} \times 4}_{\text{even}} + \frac{\overline{aabc}}{25}$$

If $\overline{bc} = 75$, then

$$\frac{\overline{aabc}}{25} = \frac{\overline{aa} \times 100 + 75}{25} = \overline{aa} \times 4 + 3,$$

so $\overline{aabc}/25$ is odd. This means $N/25$ is also odd. The digit sum will be equal to $25(2a + 2 + 5) = 25(2a + 12)$. Since N is divisible by $25(2a + 12)$, $N/25$ must be divisible by $(2a + 12)$, which is even, but this is impossible since no odd number is not divisible by an even number.Thus, $\overline{bc} \neq 75$.
If $\overline{bc} = 25$, then

$$\frac{\overline{aabc}}{25} = \frac{\overline{aa} \times 100 + 25}{25} = \overline{aa} \times 4 + 1,$$

so $\overline{aabc}/25$ is odd. $N/25$ is odd as well. The digit sum will be equal to $25(2a + 2 + 5) = 25(2a + 7)$. By using a similar argument as above, $N/25$ must be divisible by $(2a + 7)$, which is odd. Since an odd number can be divided by some other odd numbers, this might be possible. (In fact, upon performing computer verification, $N = 11251125\cdots1125$ and $N = 55255525\cdots5525$ works.)

Hence, the last two digits of N is $\boxed{25}$.

■

Problem 6 Given a triangle ABC with area 200 and perimeter 50, such that its incircle is centered at I. Point D is drawn on the triangle's circumcircle so that AD passes through I and intersects side BC at E. If $CD = 10$, find DE.

Solution

Claim: The construction of $\triangle ABC$ is impossible.

Proof: Let a, b, c be the side lengths of BC, CA, AB respectively. We are given $a+b+c = 50$ from its perimeter. Using Heron's formula,

$$\sqrt{25(25-a)(25-b)(25-c)} = 200$$
$$(25-a)(25-b)(25-c) = 1600$$

Using the AM-GM inequality,

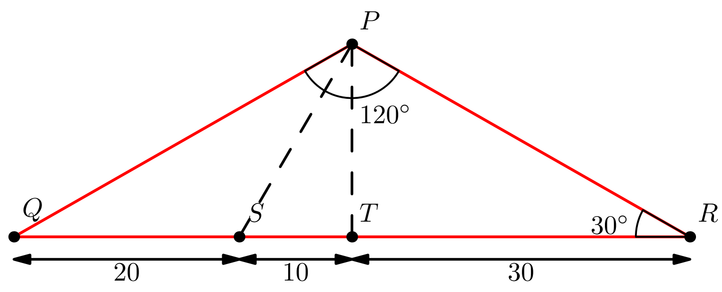
$$\sqrt[3]{(25-a)(25-b)(25-c)} \leq \frac{75-50}{3}$$
$$(25-a)(25-b)(25-c) \leq \frac{15625}{27}$$

Since the value of $(25-a)(25-b)(25-c)$ cannot exceed $15625/27$, which is lesser than 1600, this means $\triangle ABC$ cannot have both the area 200 and perimeter 50 at the same time. This problem has **no correct answer**.

■

Problem 7 Given that in triangle PQR, $PQ = PR$ and $\angle QPR = 120°$. S is a point on QR such that $QS = 20$ and $SR = 40$. Find PS.

Solution



Let T be the midpoint of QR. Notice that $PT \perp TS$ since $\triangle PQR$ is isosceles.

$$\begin{aligned} PT &= RT \tan \angle PRT \\ &= 30 \tan 30^\circ \\ &= 10\sqrt{3} \\ PS &= \sqrt{(10\sqrt{3})^2 + 10^2} \\ \therefore PS &= \boxed{20} \end{aligned}$$

■

Problem 8 What is the smallest k such that among any k consecutive positive integers, there is always a number with a digit sum (in base 10) that is divisible by 11?

Solution Source: *StackExchange - Among k consecutive numbers one has sum of digits divisible by* 11

Define $r(n)$ as the remainder when the sum of the digits of n is divided by 11. A number n is considered *good* if $r(n) = 0$, meaning that its digit sum is divisible by 11.

Claim: The maximum gap between two consecutive good numbers is 39, and this bound is optimal.

Proof: To support this claim, consider two specific numbers: $n = 999980$ and $n' = 1000019$. These two numbers are consecutive good numbers, with $n' - n = 39$, demonstrating that 39 is indeed the maximum possible gap between them.

To understand why this is the case, we define $p(n)$, the number of trailing nines in the decimal representation of n. For instance, $p(52899) = 2$ since there are two trailing nines. When we increment n by 1, the $p(n)$ trailing nines become zeros, and the digit immediately before the trailing nines increases by 1 ($899 \rightarrow 900$). This adjustment affects $r(n)$ as follows:

$$r(n+1) \equiv r(n) + 1 - 9p(n)$$

$$r(n+1) \equiv r(n) + 1 + 2p(n) \pmod{11} \tag{7}$$

Let's assume n is a good number and write n as

$$n = n_0 + j,$$

where n_0 is a multiple of 10, and j is an integer between 0 and 9. We construct sequences of numbers

$$n_s := n_0 + 10s$$

for $s \geq 1$, representing numbers in the same decade as n with respect to $r(n)$.

Now we consider the following cases:

- If $r(n_1) \neq 1$, meaning $r(n_1) \in \{0, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, then there is a good number in the decade beginning with n_1 (this can be seen when you add $0, 9, 8, 7, 6, 5, 4, 3, 2, 1$ to n_1), and $n' - n \leq 19$.

- If $r(n_1) = 1$ but $r(n_2) \neq 1$, there is a good number n' in the decade beginning with n_2, and $n' - n \leq 29$.

- The remaining case is when $r(n_1) = r(n_2) = 1$. By considering the recurrence relation (7),

$$r(n_2 - i) - r(n_2 - (i+1)) \equiv 1 + 2p(n_2 - (i+1))$$

If we take the sum on both sides from $i = 0$ to $i = 9$,

$$\begin{aligned}\sum_{i=0}^{9}(r(n_2 - i) - r(n_2 - (i+1))) &\equiv \sum_{i=0}^{9}(1 + 2p(n_2 - (i+1))) \\ r(n_2) - r(n_2 - 10) &\equiv 10 + 2p(n_2 - 1) + \sum_{i=2}^{10} 2p(n_2 - i) \pmod{11}\end{aligned}$$

Since $p(n_2 - i) = 0$ for $2 \leq i \leq 10$,

$$\begin{aligned}r(n_2) - r(n_1) &\equiv 10 + 2p(n_2 - 1) \\ 10 + 2p(n_2 - 1) &\equiv 0 \pmod{11}\end{aligned}$$

Finally, $p(n_2 - 1) \equiv 6 \pmod{11}$. Since $p(n_2 - 1) \geq 2$, this makes $p(n_3 - 1) = 1$. (For example, $n_2 - 1 = 2999 \implies n_3 - 1 = 3009$). Using (7) again,

$$\begin{aligned}r(n_3) - r(n_3 - 10) &\equiv 10 + 2p(n_3 - 1) + \sum_{i=2}^{10} 2p(n_3 - i) \\ r(n_3) - r(n_2) &\equiv 10 + 2 \\ r(n_3) &\equiv 13 \\ &\equiv 2 \pmod{11}\end{aligned}$$

$n' := n_3 + 9$ is good (since $r(n_3 + 9) \equiv 0 \pmod{11}$), so $n' - n \leq 39$ and we are done.

Therefore, the smallest k such that among any k consecutive positive integers, there is always a number with a digit sum (in base 10) that is divisible by 11 is $k = \boxed{39}$.

∎

Problem 9 Given $a^7 + a^6 + a^5 + a^4 + a^3 + a^2 + a^1 + a^0 = 0$. Find $a^{4048} + a^{2024} + 1$.

Solution The sequence $1, a, a^2, \cdots, a^7$ forms a geometric progression, so we can rewrite the given equation:

$$a^7 + a^6 + a^5 + a^4 + a^3 + a^2 + a^1 + a^0 = 0$$
$$\frac{a^8 - 1}{a - 1} = 0$$

We have $a^8 = 1$, where $a \neq 1$. The desired expression can be expressed as follows:

$$a^{4048} + a^{2024} + 1 = (a^8)^{506} + (a^8)^{251} + 1$$
$$= 1 + 1 + 1$$
$$\therefore a^{4048} + a^{2024} + 1 = \boxed{3}$$

■

Problem 10 Sean wants to create a word using the letters from 'ABC' only. He may use the same letters as much as he wants. How many 5-letter words can he make such that no two letters B are beside each other?

Solution We shall proceed with casework:

- If there are 0 Bs, then each letter of the word can be either A or C. $2^5 = 32$ words can be formed.
- If there is 1 B, there are 5 positions for B to be placed. The other 4 letters of the word can be either A or C. $5 \times 2^4 = 80$ can be formed.
- If there are 2 Bs, there are $\binom{5}{2} = 10$ ways to arrange the two Bs. If both Bs are together, then there are 4 ways to arrange the Bs. The other 3 letters of the word can be either A or C. $(10 - 4) \times 2^3 = 48$ words can be formed.
- If there are 3 Bs, there is only one arrangement such that no two Bs are together.

  $$B\_B\_B$$

  $2^2 = 4$ words can be formed.
- It is impossible to form a word such no two letters B are together if there are 4 Bs or 5 Bs.

In total, Sean can make $32 + 80 + 48 + 4 = \boxed{164}$ five-letter words.

■