SMC KMC Workshop 2024

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§1 Algebra

§1.1 Inequalities and Absolute Values

Theorem 1.1 (Trichotomy property)
Let $a, b \in \mathbb{R}$. We say that:

- a is greater than b and we write $a > b$ if and only if $a - b$ is positive.
- a is less than b and we write $a < b$ if and only if $a - b$ is negative.

Then, one and only one of the following relations is true.

$$a > b \quad \text{or} \quad a = b \quad \text{or} \quad a < b$$

This is known as the **Trichotomy property of real numbers**.

> **Remark 1.2.** The most helpful inequality theorems in high school level olympiads are the **QM-AM-GM-HM Inequality** and the **Cauchy-Schwarz Inequality**.

§1.2 Famous identities

1. **Difference of squares** $\quad a^2 - b^2 = (a-b)(a+b)$
2. **Sum/difference of cubes** $\quad a^3 \pm b^3 = (a \pm b)(a^2 \mp ab + b^2)$
3. $x^{2n} - y^{2n} = (x^n - y^n)(x^n + y^n)$
4. $x^n - y^n = (x-y)(x^{n-1} + x^{n-2}y + x^{n-3}y^2 + \cdots + xy^{n-2} + y^{n-1}), \quad \forall n \in \mathbb{Z}^+$
5. $(a+b+c)^2 = a^2 + b^2 + c^2 + 2(ab + bc + ca)$
6. $a^3 + b^3 + c^3 - 3abc = (a+b+c)(a^2 + b^2 + c^3 - ab - bc - ca)$
7. **Sophie-Germain identity** $\quad x^4 + 4y^4 = (x^2 + 2y^2 - 2xy)(x^2 + 2y^2 + 2xy)$

§1.3 Partial fractions

1. $\dfrac{1}{k(k+m)} = \dfrac{1}{m}\left(\dfrac{1}{k} - \dfrac{1}{k+m}\right)$
2. $\dfrac{1}{k(k+1)(k+2)} = \dfrac{1}{2}\left[\dfrac{1}{k(k+1)} - \dfrac{1}{(k+1)(k+2)}\right]$

§1.4 Important sums

1. **Faulhaber's formulae**
   a) $1 + 2 + \cdots + n = \dfrac{n(n+1)}{2}$
   b) $1^2 + 2^2 + \cdots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$
   c) $1^3 + 2^3 + \cdots + n^3 = \dfrac{n^2(n+1)^2}{4}$
   d) **Nicomachus's theorem** $\quad 1^3 + 2^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2$
2. **Binomial theorem** $\quad (x+y)^n = \displaystyle\sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$
3. **Multinomial theorem** $\quad (x_1 + x_2 + \cdots + x_m)^n = \displaystyle\sum_{k_1 + \cdots + k_m = n} \binom{n}{k_1, \ldots, k_m} x_1^{k_1} x_2^{k_2} \cdots x_m^{k_m}$

§1.5 Practice Problems 1

1. (Student, 2017 #23) If $|x| + x + y = 5$ and $x + |y| - y = 10$, what is the value of $x + y$?
   (A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5
2. (Student, 2018 #21) How many real solutions does the equation $||4x - 3| - 2| = 1$ have?
   (A) 2 **(B)** 3 **(C)** 4 **(D)** 5 **(E)** 6

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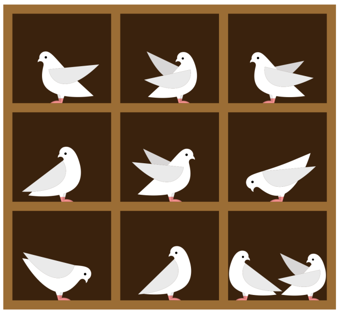
§2 Combinatorics

§2.1 Pigeonhole Principle

Figure 1: Visualisation for $m = 9$



Lemma 2.1 (Pigeonhole Principle)
When $m + 1$ pigeons enter m pigeonholes (m is a positive integer), there must be at least one hole having more than 1 pigeon.

Theorem 2.2 (Generalised Pigeonhole Principle)
When $m + 1$ pigeons enter n pigeonholes, there must be one hole having at least $\left\lfloor \frac{m}{n} \right\rfloor + 1$ pigeons.

Theorem 2.3 (Infinite Pigeonhole Principle)
When infinitely many elements are partitioned into finitely many sets, there must be at least one set containing infinitely many elements.

§2.2 Principle of Inclusion-Exclusion

PIE is essentially a special application of one of the most fundamental counting techniques: strategic overcounting. In a typical PIE computation, we will repeatedly overcount and undercount until, at the end of the process, we arrive at exactly the correct count.

Lemma 2.4

Given two sets A and B,

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Can we generalise this fact?

Theorem 2.5 (Inclusion-Exclusion Principle)

Given finite sets $A_1, A_2, \ldots, A_n$. Let S_k be the sum of cardinalities of the intersection of any k sets for $1 \le k \le n$. Then,

$$\left| \bigcup_{k=1}^{n} A_k \right| = \sum_{k=1}^{n} (-1)^{k+1} S_k.$$

§2.3 Practice Problems 2

1. (Student, 2013 #20) A box contains 900 cards numbered from 100 to 999. Any two cards have different numbers. Francois picks some cards and determines the sum of the digits on each of them. At least how many cards must he pick in order to be certain to have three cards with the same sum?
   (A) 51 **(B)** 52 **(C)** 53 **(D)** 54 **(E)** 55

2. (Student, 2015 #22) Blue and red rectangles are drawn on a blackboard. Exactly 7 of the rectangles are squares. There are 3 red rectangles more than blue squares. There are 2 red squares more than blue rectangles. How many blue rectangles are there on the blackboard?
   (A) 1 **(B)** 3 **(C)** 5 **(D)** 6 **(E)** 10

3. (Student, 2018 #23) There are 40% more girls than boys in a class. How many pupils are in this class if the probability that a two-person delegation selected at random consists of a girl and a boy is equal to 0.5?
   (A) 20 **(B)** 24 **(C)** 36 **(D)** 38 **(E)** 42

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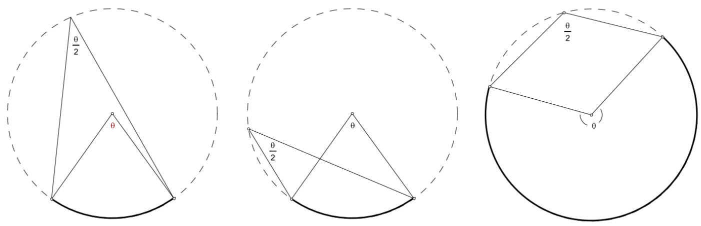
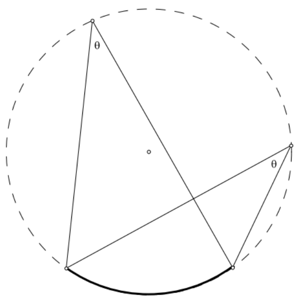
§3 Geometry

§3.1 Angles in a circle

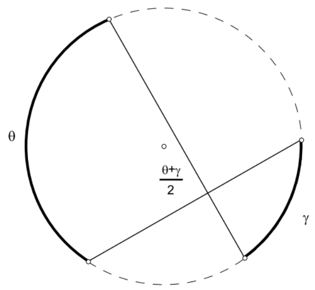
> **Remark 3.1.** It is always good to remember some angle chasing techniques too!



Lemma 3.2 (Star Trek Lemma)
An inscribed angle has half as many degrees as the intercepted arc.

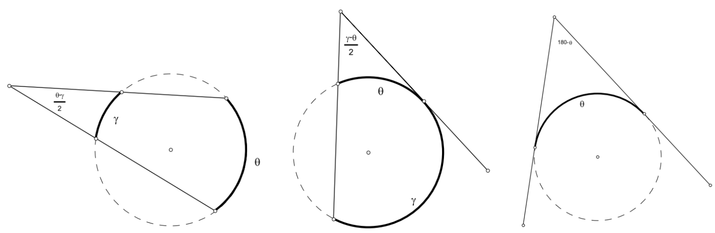


Lemma 3.3
Different inscribed angles intercepts the same arcs are equal.



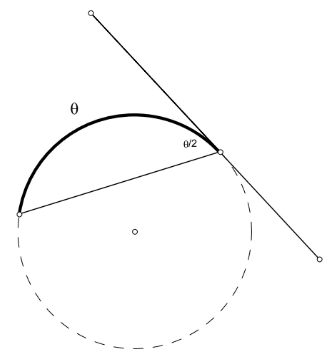
Lemma 3.4

An angle formed by two chords intersecting within a circle has one half as many degrees as the sum of intercepted arcs.



Lemma 3.5

Any angle formed by two secants, a secant and tangent, or two tangents is equal to half the different of the intercepted arcs.

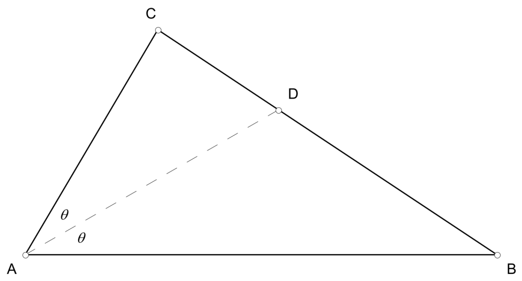


Lemma 3.6

An angle formed by a chord and a tangent to a circle has half as many degrees as the intercepted arc.

$$\frac{1}{2}\angle AOT = \angle ATB$$

§3.2 Angle Bisector Theorem



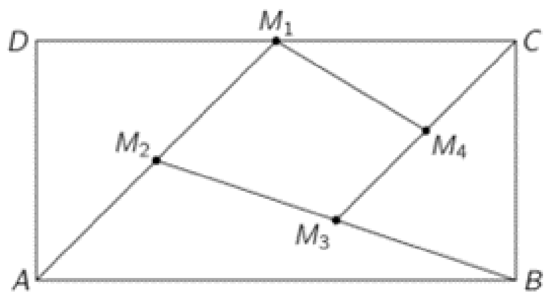
Theorem 3.7 (Angle Bisector Theorem)
If D is the point where the angle bisector of $\angle A$ in $\triangle ABC$ meets BC, then

$$\frac{BD}{BA} = \frac{CD}{CA}.$$

§3.3 Practice Problems 3

1. (Student, 2013 #23) There are some straight lines drawn on the plane. Line a intersects exactly three other lines and line b intersects exactly four other lines. Line c intersects exactly n other lines, with $n \neq 3, 4$. Determine the number of lines drawn on the plane.
   (A) 4 **(B)** 5 **(C)** 6 **(D)** 7 **(E)** 8

2. (Student, 2015 #21) In the rectangle $ABCD$ shown in the figure, M_1 is the mdipoint of CD, M_2 is the midpoint of AM_1, M_3 is the midpoint of BM_2 and M_4 is the midpoint of CM_3. Find the ratio between the areas of the quadrilateral $M_1M_2M_3M_4$ and of the rectangle $ABCD$.



   (A) $\frac{7}{16}$ **(B)** $\frac{3}{16}$ **(C)** $\frac{7}{32}$ **(D)** $\frac{9}{32}$ **(E)** $\frac{1}{5}$

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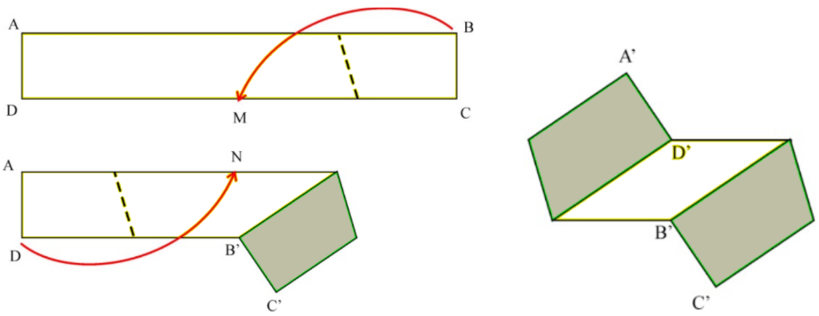
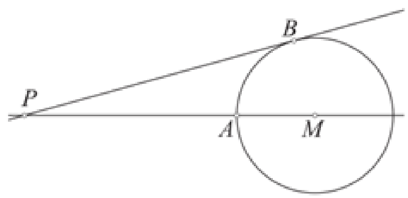
3. (Student, 2016 #22) A cube is dissected into 6 pyramids by connecting a given point in the interior of the cube with each vertex of the cube. The volumes of five of these pyramids are 2, 5, 10, 11 and 14. What is the volume of the sixth pyramid?
(A) 1 **(B)** 4 **(C)** 6 **(D)** 9 **(E)** 12

4. (Student, 2016 #23) A rectangular strip ABCD of paper 5 cm wide and 50 cm long is white on one side and grey on the other. Folding the strip, Cristina makes the vertex B coincide with the midpoint M of the side CD. Folding again, she makes the vertex D coincide with the midpoint N of the side AB.What is the area (in cm^2) of the visible white part of the strip in the last picture?



(A) 50 **(B)** 60 **(C)** 62.5 **(D)** 100 **(E)** 12

5. (Junior, 2017 #24) Points A and B are on the circle with centre M. PB is tangent to the circle at B. The distances PA and MB are integers. We know that $PB = PA + 6$. How many possible values are there for the length of MB?



(A) 0 **(B)** 2 **(C)** 4 **(D)** 6 **(E)** 8

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§4 Number Theory

§4.1 Divisibility Rules

Divisible by	Condition
2	The last digit is divisible by 2.
3	The sum of the digits must be divisible by 3.
4	The last two digits are divisible by 4.
5	The last digit is 0 or 5.
7	Forming an alternating sum of blocks of three from right to left, check if it is divisible by 7.
8	The last three digits are divisible by 8.
9	The sum of the digits are divisible by 9.
11	Forming an alternating sum of digits from left to right, check if it is divisible by 11.
13	Forming an alternating sum of blocks of three from right to left, check if it is divisible by 13.

§4.2 Practice Problems 4

1. (Student, 2014 #21) Tom wants to write several distinct positive integers, none of them exceeding 100. Their product should not be divisible by 54. At most how many integers can he write?
   (A) 8 **(B)** 17 **(C)** 68 **(D)** 69 **(E)** 90

2. (Student, 2015 #24) In the word KANGAROO, Bill and Bob replace the letters by digits, so that the resulting numbers are multiples of 11. They each replace different letters by different digits and the same letters by the same digits ($K \neq 0$). Bill obtains the largest possible such number and Bob the smallest. In both cases one of the letters is replaced by the same digit. Which digit is this?
   (A) 0 **(B)** 3 **(C)** 4 **(D)** 5 **(E)** 6

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3. (Student, 2017 #24) How many three-digit positive integers ABC exist, such that $(A+B)^C$ is a three-digit integer and an integer power of 2?
   Note: An integer power of 2 is a number in the form 2^k , where k is an integer.
   (A) 15 **(B)** 16 **(C)** 18 **(D)** 20 **(E)** 21

4. (Student, 2018 #24) Archimedes calculated 15!. The result is written on the board. Unfortunately two of the figures, the second and the tenth, are not visible. Which are these two figures?

   (A) 2 & 0 **(B)** 4 & 8 **(C)** 7 & 4 **(D)** 9 & 2 **(E)** 3 & 8

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