The Integration Bee Shortcut
(Part I: Limits)

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Contents

§1 The First Step: Limits

Before we begin with the main content of this handout, here are some prerequisites you would need to fully comprehend this material:

- High school level algebra
- Basic knowledge of elementary functions and graphs
- Trigonometric functions and identities (if you are still unfamiliar with identities, you may refer to Appendix A)
- The willpower to bear with some tedious work.

If you've got all of the above ready, we're set to go!

§1.1 Introduction to limits: definition, notation, and intuition

At its very essence, limits examine the value of a function, say $f(x)$ as its input, namely x approaches some number a. Its name speaks well for itself, the description of the limit basically says that while we are not looking for the value of f when $x = a$, we are instead seeking the value of f only for values of x that are close to a. (This is especially important when we are dealing with cases whereby the value of $f(a)$ is undefined or $f(a) \neq L$.)

We say that the function $f(x)$ approaches a number L as x approaches a, i.e. the limit of $f(x)$ as x approaches a is L, and it is expressed as:

$$\lim_{x \to a} f(x) = L$$

> **Remark 1.1.** In this handout, we will be skipping the $\epsilon - \delta$ definition of limit. You may refer to Appendix B or any university-level calculus textbook for more info.

We begin our study by examining the two following cases:

Example 1.2

Find the value of

$$\lim_{x \to 5} (x+2)^2.$$

Solution. To get a rough sense of what's happening, we choose values close to 5 from the left side (e.g. 4.9, 4.99, 4.999, 4.9999) and the right side of the limit (e.g. 5.1, 5.01, 5.001, 5.0001).

x approaches from the left side of 5		x approaches from the right side of 5	
x	$(x+2)^2$	x	$(x+2)^2$
4.9	47.61	5.1	50.41
4.99	48.8601	5.01	49.1401
4.999	48.986001	5.001	49.014001
4.9999	48.99860001	5.0001	49.00140001

Table 1: The values of the $(x+2)^2$ when x approaches 5

In this case, it is clear that the value of the limit is just $(5+2)^2 = \boxed{49}$. Direct substitution using $x = 5$ works here. □

Is there a case in which direct substitution is not applicable? Look at the example below.

Example 1.3

Compute $\lim_{x\to 1} \frac{x^2-1}{x-1}$.

Solution. Ahh.. We've finally run into a problem. It seems like plugging in $x = 1$ does not work. Try it for yourself!
The function $f(x) = \frac{x^2-1}{x-1}$ is undefined at $x = 1$. There is a 'hole' in the graph at $(1, 2)$.
Nevertheless, we can still solve the limit algebraically.
By using difference of squares,

$$\frac{x^2-1}{x-1} = \frac{(x-1)(x+1)}{x-1} = x+1$$

Hence,

$$\lim_{x\to 1} \frac{x^2-1}{x-1} = \lim_{x\to 1} (x+1) = 1+1 = \boxed{2}$$

□

Often, in the study of limits and their behaviours, we would stumble upon infinity (∞). Here's one such example:

Example 1.4

Find the limit of $\frac{1}{x}$ as $x \to 0$.

Solution. The function $f(x) = \frac{1}{x}$ grows too 'large' as x gets closer and closer to 0. In other words, the absolute value of f becomes arbitrarily large to have a limit (a fixed real number). We say that the function is **not bounded**. $\lim_{x\to 0} \frac{1}{x}$ **does not exist**. □

Exercise 1.5

What is the value of $\lim_{x\to -1} \frac{x^3+1}{x+1}$?

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§1.2 Basic limit rules: sum, product, quotient, power, and constant rules

When dealing with the constant function and identity function, Theorem 1.4 can be used to compute their limit.

Theorem 1.6 (Limits for constant and identity functions)

For any real number a and any constant c,

1. Constant function: $\lim_{x \to a} c = c$
2. Identity function: $\lim_{x \to a} x = a$

Here are a few more limit laws to follow in order to simply computation.

Theorem 1.7 (The Limit Laws)

If L, M, c, and k are real numbers and

$$\lim_{x \to c} f(x) = L \qquad \text{and} \lim_{x \to c} g(x) = M, \quad \text{then}$$

1. Sum rule: $\lim_{x \to c} f(x) + g(x) = L + M$
2. Difference rule: $\lim_{x \to c} f(x) - g(x) = L - M$
3. Constant multiple rule: $\lim_{x \to c} k \cdot f(x) = k \cdot L$
4. Product rule: $\lim_{x \to c} f(x) \cdot g(x) = L \cdot M$
5. Quotient rule: $\lim_{x \to c} \dfrac{f(x)}{g(x)} = \dfrac{L}{M}$, $M \neq 0$
6. Power rule: $\lim_{x \to c} [f(x)]^n = L^n$, $n \in \mathbb{Z}^+$
7. Root rule: $\lim_{x \to c} \sqrt[n]{f(x)} = \sqrt[n]{L} = L^{\frac{1}{n}}$, $n \in \mathbb{Z}^+$

(If n is even, we assume that $f(x) \geq 0$ for x in an interval containing c)

Example 1.8 (Basic limits)

Solve the following limits.

1. $\lim_{x \to -3} (x^2 - 13)$
2. $\lim_{x \to -2} (x^3 - 2x^2 + 4x + 8)$
3. $\lim_{x \to 2} \sqrt{3x + 3}$
4. $\lim_{x \to 2} \dfrac{2x + 5}{11 - x^3}$
5. $\lim_{x \to 0} \dfrac{x^2 - x}{x}$
6. $\lim_{x \to 1} \dfrac{2 - \sqrt{x^2 + 3}}{1 - x}$

Solution.

1. $$\begin{aligned}\lim_{x\to -3}(x^2-13) &= \lim_{x\to -3} x^2 - \lim_{x\to -3} 13\\ &= (-3)^2 - 15\\ &= \boxed{-6}\end{aligned}$$

2. $$\begin{aligned}\lim_{x\to -2}(x^3-2x^2+4x+8) &= \lim_{x\to -2} x^3 - 2\lim_{x\to -2} x^2 + 4\lim_{x\to -2} x + \lim_{x\to -2} 8\\ &= (-2)^3 - 2(-2)^2 + 4(-2) + 8\\ &= \boxed{-16}\end{aligned}$$

3. $$\lim_{x\to 2}\sqrt{3x+3} = \sqrt{3(2)+3} = \boxed{3}$$

4. $$\begin{aligned}\lim_{x\to 2}\frac{2x+5}{11-x^3} &= \frac{\lim_{x\to 2}(2x+5)}{\lim_{x\to 2}(11-x^3)}\\ &= \frac{2(2)+5}{11-2^3}\\ &= \frac{9}{3} = \boxed{3}\end{aligned}$$

5. $$\begin{aligned}\lim_{x\to 0}\frac{x^2-x}{x} &= \lim_{x\to 0}\frac{x(x-1)}{x}\\ &= \lim_{x\to 0}(x-1)\\ &= 0-1 = \boxed{-1}\end{aligned}$$

6. $$\begin{aligned}\lim_{x\to 1}\frac{2-\sqrt{x^2+3}}{1-x} &= \lim_{x\to 1}\frac{(2-\sqrt{x^2+3})\cdot(2+\sqrt{x^2+3})}{(1-x)\cdot(2+\sqrt{x^2+3})}\\ &= \lim_{x\to 1}\frac{1-x^2}{(1-x)\cdot(2+\sqrt{x^2+3})}\\ &= \lim_{x\to 1}\frac{(1-x)(1+x)}{(1-x)\cdot(2+\sqrt{x^2+3})}\\ &= \lim_{x\to 1}\frac{1+x}{2+\sqrt{x^2+3}}\\ &= \frac{1+1}{2+\sqrt{1^2+3}} = \boxed{\frac{1}{2}}\end{aligned}$$

□

Exercise 1.9

Evaluate the following limits.

1. $\displaystyle\lim_{x\to-2}\left(\frac{3x^2+4x-4}{2x^2+3x-2}\right)$
2. $\displaystyle\lim_{x\to2}\left(\frac{x+70}{9-\sqrt{11-x}}\right)$
3. $\displaystyle\lim_{x\to10}\left(\frac{\sqrt{2x-4}}{|1-x|}\right)$

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§1.3 Limits at infinity

When dealing with limits, we would often stumble across infinity and asymptotes, which make up most of the content in calculus. With that said, let's begin with a vague definition of a limit at infinity first.

Definition 1.10 (Intuitive Definition of Limit at Infinity) — Let f be a function defined on some interval (a,∞). Then

$$\lim_{x\to\infty} f(x) = L$$

means that the values of $f(x)$ can be made arbitrarily close to L by requiring x to be sufficiently large.

Similarly, if we have $\lim\limits_{x\to-\infty} g(x) = L$ where g is defined on some interval $(-\infty,a)$, this means $g(x)$ can be made arbitrarily close to L by when x is a sufficiently large negative value.

Then, suppose we have $f(x) = 1/x$. From observation, we can see that when the denominator or x gets larger and larger, the fraction $1/x$ or $f(x)$ becomes smaller and smaller. By examining the graph, we see that there is a horizontal asymptote at $y=0$. This occurs because a priori there is no tangible value of x such that $f(x) = 1/x = 0$. But when we take the limit as $x\to\pm\infty$, we can see that $f(x)$ becomes infinitesimally small that it approaches 0.

If we changed $f(x)$ to $f(x) = 1/x^2, 1/x^3, \cdots$, we would notice the same thing when $x\to\pm\infty$, $f(x)\to 0$. In general,

Theorem 1.11

If $r > 0$ is a rational number, then

$$\lim_{x \to \infty} \frac{1}{x^r} = 0.$$

If $r > 0$ is a rational number such that x^r is defined for all x, then

$$\lim_{x \to -\infty} \frac{1}{x^r} = 0.$$

Example 1.12

Evaluate

$$\lim_{x \to \infty} \frac{3x^2 - x - 2}{5x^2 + 4x + 1}.$$

Solution. If we perform direct substitution, we would end up with $\frac{\infty}{\infty}$ as our answer, but this is an indeterminate form.

Instead, when approaching limits for fractions, we want to change every single term into reciprocals. Thus, we can try **dividing the numerator and denominator by the highest power of x in the denominator**.

$$\begin{aligned}
\lim_{x \to \infty} \frac{3x^2 - x - 2}{5x^2 + 4x + 1} &= \lim_{x \to \infty} \frac{3 - \frac{1}{x} - \frac{2}{x^2}}{5 + \frac{4}{x} + \frac{1}{x^2}} \\
&= \lim_{x \to \infty} \frac{3 - 0 - 0}{5 + 0 + 0} \\
&= \boxed{\frac{3}{5}}.
\end{aligned}$$

□

Example 1.13

Solve the limit

$$\lim_{x \to \infty} \left(\sqrt{x^2 + 2} - x \right).$$

Then, solve

$$\lim_{x \to -\infty} \left(\sqrt{x^2 + 2} - x \right).$$

Solution. Direct substitution will yield the indeterminate form $\infty - \infty$. To proceed, we rationalise the numerator.

$$\begin{aligned}\lim_{x\to\infty}\left(\sqrt{x^2+2}-x\right) &= \lim_{x\to\infty}\frac{\left(\sqrt{x^2+2}-x\right)\left(\sqrt{x^2+2}+x\right)}{\left(\sqrt{x^2+2}+x\right)}\\ &= \lim_{x\to\infty}\frac{(x^2+2)-x^2}{\left(\sqrt{x^2+2}+x\right)}\\ &= \lim_{x\to\infty}\frac{2}{\left(\sqrt{x^2+2}+x\right)}\\ &= \boxed{0}.\end{aligned}$$

What happens when x approaches $-\infty$ this time?

$$\begin{aligned}\lim_{x\to-\infty}\left(\sqrt{x^2+2}-x\right) &= \lim_{x\to-\infty}\sqrt{x^2+2}-\lim_{x\to-\infty}x\\ &= +\infty-(-\infty)\\ &= +\infty\end{aligned}$$

In other words, the limit does not exist.

> **Remark 1.14.** Note that in the case of the second limit, we skipped the rationalisation of the numerator unlike the first limit where we managed to get an indeterminate form, because we can immediately deduce that $\left(\sqrt{x^2+2}-x\right)$ is not bounded at $x=-\infty$.

□

Exercise 1.15

Determine the existence of the limits below.

1. $\lim\limits_{x\to\infty}(x^2-x)$
2. $\lim\limits_{x\to\infty}\dfrac{x^2+x}{3-x}$

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Exercise 1.16

Find the limit or show that it does not exist.

1. $\lim\limits_{x\to-\infty}\dfrac{\sqrt{9x^6-x}}{x^3+1}$
2. $\lim\limits_{x\to\infty}\arctan(\exp(x))$, (Note: $\arctan(\cdot)=\tan^{-1}(\cdot)$, $\exp(x)=e^x$)
3. $\lim\limits_{x\to\infty}\dfrac{\sin^2 x}{x^2+1}$
4. $\lim\limits_{x\to\infty}[\ln(2+x)-\ln(1+x)]$

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§1.4 One-sided limits: definition, notation, and interpretation

The existence of $\lim_{x\to a} f(x)$ depends on the behaviour of f on both sides of a.

Definition 1.17 (One-sided limits) — Let

$$\lim_{x\to a^-} f(x) = L_1$$

be the limit of $f(x)$ as x approaches from the left side of a (x is sufficiently close to a, where $x < a$), and

$$\lim_{x\to a^+} f(x) = L_2$$

be the limit of $f(x)$ as x approaches from the right side of a (x is sufficiently close to a, where $x > a$).

Theorem 1.18

$$\lim_{x\to a} f(x) = L \Leftrightarrow \lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = L.$$

Example 1.19

Find the limits below or show that they do not exist if necessary.

1. $\lim\limits_{x\to 3^-} \sqrt{x^2-9}$
2. $\lim\limits_{x\to 1^+} \dfrac{|x-1|}{x-1}$

Solution.

1. $\lim\limits_{x\to 3^-} \sqrt{x^2-9}$
   $x \to 3^-$ means $x < 3 \implies x^2 < 9 \Leftrightarrow x^2 - 9 < 0.$
   Since $\sqrt{x^2-9}$ is undefined for $x < 3$, $\lim\limits_{x\to 3^-} \sqrt{x^2-9}$ does not exist.

2. $\lim\limits_{x\to 1^+} \dfrac{|x-1|}{x-1}$ $x \to 1^+$ means $x > 1$
   $|x-1| = x-1$, when $x > 1$.

$$\therefore \lim_{x\to 1^+} \frac{|x-1|}{x-1} = \lim_{x\to 1^+} \frac{x-1}{x-1} = 1$$

$\square$

Exercise 1.20

The function f is defined by

$$f(x) = \begin{cases} 5 + ax & x < 1 \\ x^2 + b & 1 \le x < 6 \\ x^2 - 2 & x \ge 6 \end{cases}$$

If $\lim_{x\to 1} f(x)$ and $\lim_{x\to 6} f(x)$ exist, find the values of a and b.

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§1.5 The Squeeze Theorem

Before I present you with the Squeeze Theorem, we should get to know how inequalities work with limits.

Theorem 1.21

If $f(x) \le g(x)$ when x is near a (except possibly at a) and the limits of f and g both exist as x approaches a, then

$$\lim_{x\to a} f(x) \le \lim_{x\to a} g(x).$$

Remark 1.22. The assertion resulting fro replacing the less than or equal to ($\le$) inequality by the strict less ($<$) inequality in the theorem above is false.

The Squeeze Theorem, otherwise known as the Sandwich Theorem or the Pinching Theorem states the following:

Theorem 1.23 (The squeeze theorem)

If $g(x) \le f(x) \le h(x)$ in an open interval containing c, except possibly at $x = c$, and if

$$\lim_{x\to c} g(x) = \lim_{x\to c} h(x) = L$$

then $\lim\limits_{x\to c} f(x) = L$.

Example 1.24

Given that

$$1 - \frac{x^2}{4} \le u(x) \le 1 + \frac{x^2}{2}$$

for all $x \ne 0$, find $\lim_{x\to 0} u(x)$, no matter how complicated u is.

Solution. Since

$$\lim_{x\to 0}\left(1 - \frac{x^2}{4}\right) = 1 \text{ and } \lim_{x\to 0}\left(1 + \frac{x^2}{2}\right) = 1,$$

the Squeeze Theorem implies that

$$\lim_{x\to 0} u(x) = \boxed{1}$$

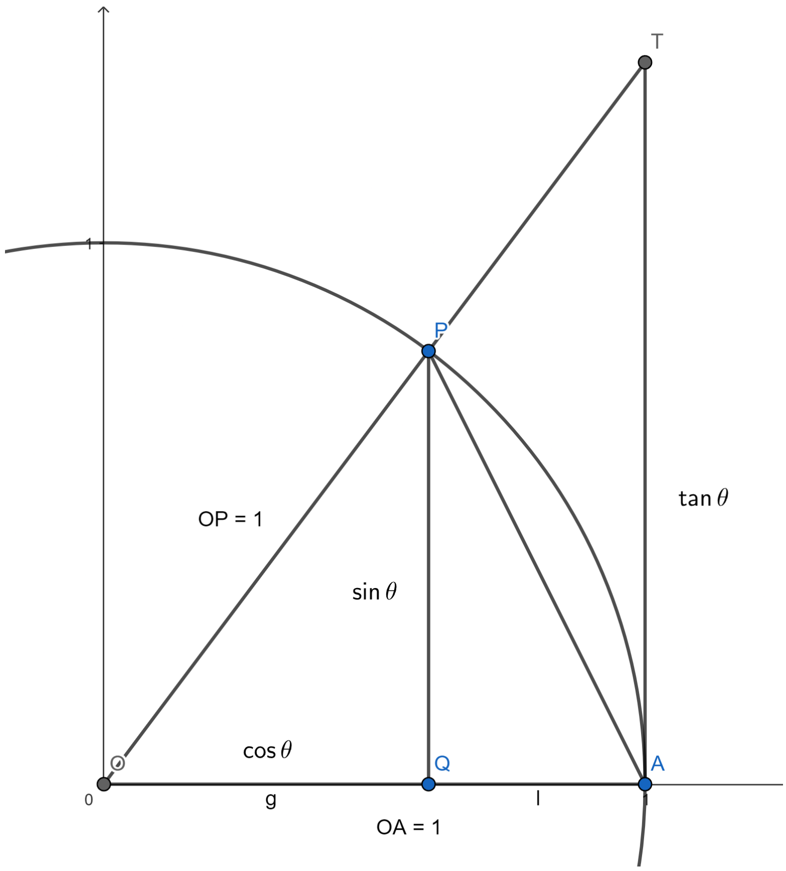
□

Another important application of the Squeeze Theorem involves finding the important limit of the ratio $\sin\theta/\theta$ as $\theta \to 0$.

Example 1.25

Show that

$$\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1 \qquad (\theta \text{ in radians}).$$



Solution. We begin by showing that the right-hand limit is 1, so $0 < \theta < \frac{\pi}{2}$. Notice that

$$[\triangle OAP] < \text{ area sector } OAP < [\triangle OAT]$$

We can express these area in terms of θ as follows:

$$\begin{aligned}
[\triangle OAP] &= \frac{1}{2}(1)(\sin\theta) = \frac{1}{2}\sin\theta \\
\text{Area sector } OAP &= \frac{1}{2}(1)^2\theta = \frac{\theta}{2} \\
[\triangle OAT] &= \frac{1}{2}(1)(\tan\theta) = \frac{1}{2}\tan\theta
\end{aligned}$$

Thus,

$$\frac{1}{2}\sin\theta < \frac{\theta}{2} < \frac{1}{2}\tan\theta.$$

This last inequality goes the same way if we divide all terms by $\frac{1}{2}\sin\theta$.

$$1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}$$

$$1 > \frac{\sin\theta}{\theta} > \cos\theta$$

Since $\lim_{\theta\to 0^+}\cos\theta = 1$, the Squeeze Theorem gives

$$\lim_{\theta\to 0^+}\frac{\sin\theta}{\theta} = 1.$$

To consider the left-hand limit, we recall that $\sin\theta$ and θ are both odd functions. Therefore, $\frac{\sin\theta}{\theta}$ is an even function (graph is symmetric about y-axis).

$$\therefore \lim_{\theta\to 0^-}\frac{\sin\theta}{\theta} = \lim_{\theta\to 0^+}\frac{\sin\theta}{\theta} = 1 \Leftrightarrow \lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1.$$

Remark 1.26. It would be inappropriate to apply L'Hopital's rule to derive the limit above since differentiating $\sin\theta$ would require us to know this limit beforehand. Using this method is equivalent to circular reasoning.

□

Exercise 1.27

Show that

$$\lim_{x\to 0} x^2 \sin\frac{1}{x} = 0.$$

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Exercise 1.28

Find the perimeter of a regular n-gon inscribed in a unit circle. What value does this perimeter approach when n becomes very large?

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§1.6 L'Hopital's rule (optional)

Before you proceed with this section, it is strongly advised that you read Chapter 2 first as L'Hopital's rule would involve differentiation to solve special limits.

Theorem 1.29 (L'Hopital's rule)

For functions f and g which are differentiable on an open interval I, except possibly at a point c contained in I, if

- $\lim\limits_{x\to c} f(x) = \lim\limits_{x\to c} g(x) = 0$ or $\lim\limits_{x\to c} f(x) = \pm\infty$ and $\lim_{x\to c} g(x) \pm \infty$, and;
- $g'(x) \neq 0$ for all x in I with $x \neq c$ and;
- $\lim\limits_{x\to c} \dfrac{f'(x)}{g'(x)}$ exists,

then

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}.$$

§1.7 Practice Problems 1

Challenging limits

1. Find
$$\lim_{x\to\infty} \left(\sqrt{x^2+3x} - x\right).$$

2. Show that
$$\lim_{n\to\infty} \left(1 + \frac{a}{n}\right)^{nb} = e^{ab}.$$

3. (Chen Jing Run Senior Sample) Find
$$\lim_{n\to\infty} \left(\frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \frac{1}{\sqrt{n^2+3}} + \cdots + \frac{1}{\sqrt{n^2+2n}}\right).$$

4. (Hualo Geng Senior 2018) Find the value of
$$\lim_{n\to\infty} \left(\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}\right).$$

5. Evaluate
$$\lim_{n\to\infty} \left(\frac{1}{\sqrt{n}\sqrt{n+1}} + \frac{1}{\sqrt{n}\sqrt{n+2}} + \cdots + \frac{1}{\sqrt{n}\sqrt{2n}}\right).$$

6. Find the value of
$$\lim_{n\to\infty} \left(\frac{n!}{n^n}\right)^{\frac{1}{n}}.$$

§1.8 Solutions for Chapter 1

Chapter exercises (work in progress)
Challenging limits (Practice Problems 1)

1. We can approach the problem with two methods.

Method 1: Limit growth
By completing the square, we obtain

$$\begin{aligned}\lim_{x\to\infty}(\sqrt{x^2+3x}-x) &= \lim_{x\to\infty}\left(\sqrt{\left(x^2+3x+\frac{9}{4}\right)-\frac{9}{4}}-x\right)\\ &= \lim_{x\to\infty}\left(\sqrt{\left(x+\frac{3}{2}\right)^2-\frac{9}{4}}-x\right)\end{aligned}$$

As $x\to\infty$, we see that $-9/4$ becomes insignificant compared to $(x+\frac{3}{2})^2$ term (rapid growth).
Since $(x+\frac{3}{2})^2 >> \frac{9}{4}$, our original limit is now equivalent to

$$\lim_{x\to\infty}\left(\sqrt{\left(x+\frac{3}{2}\right)^2}-x\right)=\boxed{\frac{3}{2}}.$$

Method 2: Multiplication by conjugate

$$\begin{aligned}\lim_{x\to\infty}(\sqrt{x^2+3x}-x) &= \lim_{x\to\infty}\frac{(\sqrt{x^2+3x}-x)(\sqrt{x^2+3x}+x)}{(\sqrt{x^2+3x}+x)}\\ &= \lim_{x\to\infty}\frac{x^2+3x-x^2}{(\sqrt{x^2+3x}+x)}\\ &= \lim_{x\to\infty}\frac{3}{(\sqrt{1+\frac{3}{x}}+1)}\\ &= \boxed{\frac{3}{2}}.\end{aligned}$$

2. Let $L := \lim_{n\to\infty}\left(1+\frac{a}{n}\right)^{nb}$

$$\begin{aligned}\ln L &= \ln\left(\lim_{n\to\infty}\left(1+\frac{a}{n}\right)^{nb}\right)\\ &= \lim_{n\to\infty}\ln\left(\left(1+\frac{a}{n}\right)^{nb}\right)\\ &= \frac{\lim_{n\to\infty}\left(\ln\left(1+\frac{a}{n}\right)\right)}{\lim_{n\to\infty}\frac{1}{nb}}\end{aligned}$$

Direct substitution of $n \to \infty$ for $\dfrac{\lim_{n\to\infty}\left(\ln\left(1+\frac{a}{n}\right)\right)}{\lim_{n\to\infty}\frac{1}{nb}}$ would give us $\frac{0}{0}$ indeterminate case. We can proceed by using L'Hopital's rule.

$$\begin{aligned}
\ln L &= \lim_{n\to\infty} \frac{\frac{d}{dn}\left[\ln\left(1+\frac{a}{n}\right)\right]}{\frac{d}{dn}\left[\frac{1}{nb}\right]} \\
&= \lim_{n\to\infty}\left(\frac{\frac{1}{1+\frac{a}{n}}\cdot\left(-\frac{a}{n^2}\right)}{-\frac{1}{bn^2}}\right) \\
&= \lim_{n\to\infty}\frac{ab}{1+\frac{a}{n}} \\
&= ab \\
\therefore L &= e^{ab}
\end{aligned}$$

□

3. This problem may seem daunting at first because it has an infinite sum. However, the sandwich theorem simplifies the problem tremendously.

   Let $L = \lim\limits_{n\to\infty}\sum\limits_{k=1}^{2n}\dfrac{1}{\sqrt{n^2+k}}$.
   Notice that

   $$\frac{2n}{\sqrt{n^2+2n}} \le \sum_{k=1}^{2n}\frac{1}{\sqrt{n^2+k}} \le \frac{2n}{\sqrt{n^2}}.$$

   Hence,

   $$\lim_{n\to\infty}\frac{2n}{\sqrt{n^2+2n}} \le L \le \lim_{n\to\infty}\frac{2n}{\sqrt{n^2}}$$

   $$\lim_{n\to\infty}\frac{2}{\sqrt{1+\frac{2}{n}}} \le L \le \lim_{n\to\infty}\frac{2n}{n}$$

   $$2 \le L \le 2$$

   $$\therefore L = \lim_{n\to\infty}\left(\frac{1}{\sqrt{n^2+1}}+\frac{1}{\sqrt{n^2+2}}+\frac{1}{\sqrt{n^2+3}}+\cdots+\frac{1}{\sqrt{n^2+2n}}\right) = \boxed{2}.$$

4. Let $S_n = \frac{1}{n+1}+\cdots+\frac{1}{2n}$. By considering areas of rectangles of height $\frac{1}{n+k}$ and base 1, we see

   $$S_n < \int_n^{2n}\frac{dx}{x} = \ln 2n - \ln n = \ln 2.$$

   Similarly,

   $$\int_{n+1}^{2n+1}\frac{dx}{x} = \ln(2n+1)-\ln(n+1) < S_n.$$

   When we take the limit for the inequality,

   $$\lim_{n\to\infty}\left(\ln(2n+1)-\ln(n+1)\right) \le \lim_{n\to\infty} S_n \le \ln 2$$

$$\lim_{n\to\infty} \ln\left(2 - \frac{1}{n+1}\right) \leq \lim_{n\to\infty} S_n \leq \ln 2$$

By the Squeeze Theorem, our desired answer is

$$\lim_{n\to\infty} S_n = \boxed{\ln 2}$$

5. The plan is to evaluate this limit using the Riemann sum definition of a definite integral.
   Let $L := \lim_{n\to\infty}\left(\frac{1}{\sqrt{n}\sqrt{n+1}} + \frac{1}{\sqrt{n}\sqrt{n+2}} + \cdots + \frac{1}{\sqrt{n}\sqrt{2n}}\right)$.

   We can rewrite L as follows

$$\begin{aligned} L &= \lim_{n\to\infty}\left(\sum_{i=1}^{n} \frac{1}{\sqrt{n}\sqrt{n+i}}\right) \\ &= \lim_{n\to\infty}\left(\sum_{i=1}^{n} \frac{1}{\sqrt{n}} \cdot \frac{1}{\sqrt{n\left(1+\frac{i}{n}\right)}}\right) \\ &= \lim_{n\to\infty}\left(\sum_{i=1}^{n} \frac{1}{n} \cdot \frac{1}{\sqrt{1+\frac{i}{n}}}\right) \end{aligned}$$

   After change of variables $n \to x$, let $\Delta x = \frac{1}{x}$, $x_i = i\Delta x = \frac{i}{x}$ and $f(x) = \frac{1}{\sqrt{1+x}}$. As $\Delta x \to 0$, we obtain

$$L = \int_0^1 \frac{1}{\sqrt{1+x}}\, dx = \boxed{2\sqrt{2} - 2}$$

6. Let $L := \lim_{n\to\infty}\left(\frac{n!}{n^n}\right)^{\frac{1}{n}}$

$$\begin{aligned} \ln L &= \ln \lim_{n\to\infty}\left(\frac{n!}{n^n}\right)^{\frac{1}{n}} \\ &= \lim_{n\to\infty}\left(\frac{1}{n}\ln\left(\frac{n!}{n^n}\right)\right) \\ &= \lim_{n\to\infty}\left(\frac{1}{n}\sum_{k=1}^{n}\ln\left(\frac{k}{n}\right)\right) \end{aligned}$$

   After change of variables $n \to x$, let $\Delta x = \frac{1}{x}$, $x_k = k\Delta x = \frac{k}{x}$ and $f(x) = \ln x$. As

$\Delta x \to 0$, we obtain

$$\begin{aligned}
\ln L &= \int_0^1 \ln x \, dx \\
&= [x \ln |x| - x]_0^1 \\
&= (-1) - \left(\lim_{x \to 0^+} x \ln x - x \right), \qquad x > 0 \implies |x| = x \\
&= -1 - \left(\lim_{x \to 0^+} \frac{\ln x - 1}{\frac{1}{x}} \right) \\
&\overset{\text{L.H.}}{=} -1 - \left(\lim_{x \to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} \right) \\
&= -1 - \left(\lim_{x \to 0^+} -x \right) \\
&= -1 \\
L &= \boxed{\frac{1}{e}}
\end{aligned}$$

Appendix A: Important Algebra Stuff

Famous identities

1. **Difference of squares** $a^2 - b^2 = (a-b)(a+b)$
2. **Sum/difference of cubes** $a^3 \pm b^3 = (a \pm b)(a^2 \mp ab + b^2)$
3. $x^{2n} - y^{2n} = (x^n - y^n)(x^n + y^n)$
4. $x^n - y^n = (x-y)(x^{n-1} + x^{n-2}y + x^{n-3}y^2 + \cdots + xy^{n-2} + y^{n-1}), \quad \forall n \in \mathbb{Z}^+$
5. $(a+b+c)^2 = a^2 + b^2 + c^2 + 2(ab + bc + ca)$
6. $a^3 + b^3 + c^3 - 3abc = (a+b+c)(a^2 + b^2 + c^3 - ab - bc - ca)$
7. **Sophie-Germain identity** $x^4 + 4y^4 = (x^2 + 2y^2 - 2xy)(x^2 + 2y^2 + 2xy)$

Partial fractions

1. $\dfrac{1}{k(k+m)} = \dfrac{1}{m}\left(\dfrac{1}{k} - \dfrac{1}{k+m}\right)$
2. $\dfrac{1}{k(k+1)(k+2)} = \dfrac{1}{2}\left[\dfrac{1}{k(k+1)} - \dfrac{1}{(k+1)(k+2)}\right]$

Important sums

1. **Faulhaber's formulae**
   a) $1 + 2 + \cdots + n = \dfrac{n(n+1)}{2}$
   b) $1^2 + 2^2 + \cdots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$
   c) $1^3 + 2^3 + \cdots + n^3 = \dfrac{n^2(n+1)^2}{4}$
   d) **Nicomachus's theorem** $1^3 + 2^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2$
2. **Binomial theorem** $(x+y)^n = \displaystyle\sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$
3. **Multinomial theorem** $(x_1 + x_2 + \cdots + x_m)^n = \displaystyle\sum_{k_1 + \cdots + k_m = n} \binom{n}{k_1, \ldots, k_m} x_1^{k_1} x_2^{k_2} \cdots x_m^{k_m}$