INTEGRATION BEE

A cheatsheet

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Some stuff to know

1 Prerequisites

1. Trigonometric identities: Addition of angles, double-angle formulae, half-angle formulae, Pythagorean theorem, sum-to-product, product-to-sum, adding $\pi/2$, π, 2π to angles.

2. Concept of limits

3. Limit Laws: Sum rule, difference rule, constant multiple rule, product rule, quotient rule, power rule, root rule.

4. Basic limit tricks:

   (a) For fractions, try dividing the numerator and denominator by the highest power of x in the denominator.

   (b) If there are expressions which involve radicals like $a + \sqrt{b}$ or $\sqrt{a} + \sqrt{b}$, multiply the numerator and denominator by the conjugate, i.e. $a - \sqrt{b}$ or $\sqrt{a} - \sqrt{b}$.

5. L' Hopital's Rule

   For functions f and g which are differentiable on an open interval I except possibly at a point c contained in I, if $\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = 0$ or $\pm\infty$ and $g'(x) \neq 0$ for all x in I with $x \neq c$, and $\lim_{x\to c} \frac{f'(x)}{g'(x)}$ exists, then

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}$$

6. Squeeze theorem

   If $g(x) \leq f(x) \leq h(x)$ in an open interval containing c, except possibly at $x = c$, and if

$$\lim_{x\to c} g(x) = \lim_{x\to c} h(x) = L,$$

   then $\lim_{x\to c} f(x) = L$.

7. Important inequalities

   (a) If $f(x) \leq g(x)$ in an open interval containing c, except possibly at $x = c$, and both limits exist, then

   $$\lim_{x \to c} f(x) \leq \lim_{x \to c} g(x).$$

   (b) For any angle θ in radians,

   $$-|\theta| \leq \sin\theta \leq |\theta|$$

   and

   $$-|\theta| \leq 1 - \cos\theta \leq |\theta|.$$

8. An important limit

   $$\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$$

2 Differentiation

1. The definition of derivative

2. General formulae
   Assume u and v are differentiable functions.

Constant:	$\frac{d}{dx}(c) = 0$
Sum:	$\frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx}$
Constant multiple:	$\frac{d}{dx}(cu) = c\frac{du}{dx}$
Product:	$\frac{d}{dx}(uv) = d\frac{dv}{dx} + v\frac{du}{dx}$
Quotient:	$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v(du/dx) - u(dv/dx)}{v^2}$
Power:	$\frac{d}{dx}x^n = nx^{n-1}$
Chain rule:	$\frac{d}{dx}(f(g(x))) = f'(g(x)) \cdot g'(x)$

3. Trigonometric functions

$$\frac{d}{dx}(\sin x) = \cos x \qquad \frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\cot x) = -\csc^2 x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x \qquad \frac{d}{dx}(\csc x) = -\csc x \cot x$$

4. Exponential and logarithmic functions

$$\frac{d}{dx}e^x = e^x \qquad \frac{d}{dx}\ln x = 1/x$$

5. Inverse trigonometric functions

$$\frac{d}{dx}(\arcsin x) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\arccos x) = -\frac{1}{\sqrt{1-x^2}}$$
$$\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2} \qquad \frac{d}{dx}(\operatorname{arccot} x) = -\frac{1}{1+x^2}$$
$$\frac{d}{dx}(\operatorname{arcsec} x) = \frac{1}{|x|\sqrt{x^2-1}} \qquad \frac{d}{dx}(\operatorname{arccsc} x) = -\frac{1}{|x|\sqrt{x^2-1}}$$

6. Hyperbolic functions

$$\frac{d}{dx}(\sinh x) = \cosh x \qquad \frac{d}{dx}(\cosh x) = \sinh x$$
$$\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x \qquad \frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x$$
$$\frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x \qquad \frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x$$

7. Inverse hyperbolic functions

$$\frac{d}{dx}(\operatorname{arsinh} x) = \frac{1}{\sqrt{1+x^2}} \qquad \frac{d}{dx}(\operatorname{arcosh} x) = \frac{1}{\sqrt{x^2-1}}$$
$$\frac{d}{dx}(\operatorname{artanh} x) = \frac{1}{1-x^2} \qquad \frac{d}{dx}(\operatorname{arsech} x) = -\frac{1}{x\sqrt{1-x^2}}$$
$$\frac{d}{dx}(\operatorname{arcoth} x) = \frac{1}{1-x^2} \qquad \frac{d}{dx}(\operatorname{arcsch} x) = -\frac{1}{x\sqrt{1-x^2}}$$

3 Integration

1. Definition of integration by Riemann sum
2. Methods of integration
   (a) Trigonometric substitution
   (b) Hyperbolic substitution
   (c) u-substitution
   (d) Splitting into partial fractions
   (e) Integration by parts (DI method)
3. Methods of numerical integration
   (a) Midpoint method
   (b) Trapezium rule
   (c) Simpson's rule

4 Basic integration tricks

1. Reflections (King's property)

   For a function f and real numbers $a < b$,

$$\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx.$$

2. Inversion

   Suppose the function f has a bounded antiderivative on $[0, \infty)$. Then, via the u-substitution $x \to 1/x$,

$$\int_0^\infty f(x)\,dx = \int_0^\infty f(x) + \frac{f(1/x)}{x^2}\,dx.$$

3. Inverse functions

   (a) If F is an antiderivative of f,

$$\int f^{-1}(x)dx = xf^{-1}(x) - F(f^{-1}(x)) + c.$$

   (b) Suppose the function f is one-to-one and increasing.

$$\int_a^b f(x)\,dx + \int_{f(a)}^{f(b)} f^{-1}(x)\,dx = bf(b) - af(a).$$

   (c) Suppose the function f is one-to-one and decreasing. Then, a geometric equivalence may be established:

$$\int_a^b f(x)\,dx - \int_{f(a)}^{f(b)} f^{-1}(x)\,dx = (b-a)f(b) - a(f(a) - f(b)).$$

4. Odd and even functions

   (a) Since an odd function o satisfies $o(-x) = -o(x)$, for a finite t,

$$\int_{-t}^t o(x)\,dx = 0.$$

   (b) An even function e satisfies $e(x) = e(-x)$, so for any t,

$$\int_{-t}^t e(x)\,dx = 2\int_0^t e(x)\,dx$$

   (c) Suppose the functions e and o are even and odd respectively, and for any function t,

$$\int_{-a}^a \frac{e(x)}{1 + t(x)^{o(x)}}\,dx = \int_0^a e(x)\,dx$$

5. Weierstrauss substitution

   By using the substitution $t = \tan x/2$, where $-\pi < x < \pi$. Then

$$\sin\left(\frac{x}{2}\right) = \frac{t}{\sqrt{1+t^2}} \qquad \text{and} \qquad \cos\left(\frac{x}{2}\right) = \frac{1}{\sqrt{1+t^2}}.$$

   Hence,

$$\sin x = \frac{2t}{1+t^2}, \qquad \cos x = \frac{1-t^2}{1+t^2}, \qquad \text{and} \qquad dx = \frac{2}{1+t^2}\,dt.$$

6. Reverse product rule and reverse quotient rule
   If u and v are differentiable functions, then

   (a) (Product rule)

$$\int udv + vdu = uv$$

   (b) (Quotient rule)

$$\int \frac{vdu - udv}{v^2} = \frac{u}{v}$$

7. Taylor series: Memorise expansions for common functions like $\frac{1}{1-x}$, $\frac{1}{1+x}$, $\sin x$, $\cos x$, e^x, $\ln(1+x)$, $\arctan x$.

8. Integration by reduction formula

9. Well-known definite integrals

   (a) Frullani integral

   If f is a function defined for all nonnegative real numbers that has a limit at ∞, which we denote by $f(\infty)$,

$$\int_0^\infty \frac{f(ax) - f(bx)}{x}\,\mathrm{d}x = \Big(f(\infty) - f(0)\Big)\ln\frac{a}{b}.$$

   (b) Dirichlet integral

$$\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}.$$

   (c) Sophomore's dream

   i.

$$\int_0^1 x^{-x}\,dx = \sum_{n=1}^{\infty} n^{-n} = 1.291285997\ldots$$

   ii.

$$\int_0^1 x^x\,dx = \sum_{n=1}^{\infty} (-1)^{n+1} n^{-n} = -\sum_{n=1}^{\infty} (-n)^{-n} = 0.7834305107\ldots$$

   (d) Gaussian integral

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}.$$

5 Multivariable and beyond

- Differentiating under the integral (Feynman's trick)
- Changing to double integrals
- Contour integration
- Laplace transform

  The Laplace transform is an integral transform that converts a function of a real variable to a function of complex variable s.
  For suitable functions f, the Laplace transform is the integral

$$\mathcal{L}\{f\}(s) = \int_0^\infty f(t)e^{-st}\,dt.$$

- Schwinger parametrization
- Harmonic functions

  Suppose f is a bivariate harmonic function, (a, b) is a point on the plane, and r is a positive real number. Then,

$$\int_0^{2\pi} f(a + r\cos\theta, b + r\sin\theta)\,d\theta = 2\pi ab$$

6 Nonelementary integrals and special functions

- Lambert W function: The inverse relation of $f(w) = we^w$, where w is any complex number.
  Properties of Lambert W:

  1. $W(xe^x) = x$
  2. $W(x)e^{W(x)} = x$
  3. $\frac{dW}{dz} = \frac{1}{z+e^{W(x)}}$ for $z \notin \left\{0, \frac{1}{e}\right\}$
  4. $\int W(x)\,dx = xW(x) - x + e^{W(x)} + C = x\left(W(x) - 1 + \frac{1}{W(x)}\right) + C.$

- Dirac delta function:

$$\delta(x) = \begin{cases} +\infty, & x = 0 \\ 0, & x \neq 0 \end{cases}$$

- Step function
- Sign function
- Trigonometric integrals (Sine integral & cosine integral)
- Exponential integral
- Logarithmic integral
- Elliptic integral

- Fresnel integral
- Gamma and pi function
- Beta function
- Riemann zeta function
- Dirichlet beta and eta function